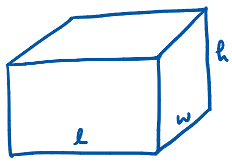
Example 3. A rectangular box is to be made from 100 m^2 of cardboard. Find the maximum volume of such a box.



We want to maximize $V = lwh$ (1) where

$2lw + 2lh + 2wh = 100$ (2) and $l, w, h > 0$.

Solve for h in (2): $h = \dfrac{100 - 2lw}{2l + 2w} = \dfrac{50 - lw}{l + w}$

Sub into (1): $V = lw\left(\dfrac{50 - lw}{l + w}\right) = \dfrac{50lw - l^2w^2}{l + w}$ ← define as $f(l, w)$

Find critical points:

$$f_l(l, w) \underset{\text{quotient rule}}{=} \frac{(l+w)(50w - 2lw^2) - (50lw - l^2w^2)(1)}{(l+w)^2} \underset{\text{some messy algebra}}{=} \frac{w^2(50 - l^2 - 2lw)}{(l+w)^2}$$

$$f_w(l, w) = \frac{(l+w)(50l - 2l^2w) - (50lw - l^2w^2)(1)}{(l+w)^2} = \frac{l^2(50 - w^2 - 2lw)}{(l+w)^2}$$

$$\left.\begin{aligned} w^2(50 - l^2 - 2lw) &= 0 \;(3) \\ l^2(50 - w^2 - 2lw) &= 0 \;(4) \end{aligned}\right\} \Rightarrow \begin{aligned} (3) &\Rightarrow \cancel{w = 0}\;(5) \text{ or } l^2 + 2lw - 50 = 0\;(6) \\ (4) &\Rightarrow \cancel{l = 0}\;(7) \text{ or } w^2 + 2lw - 50 = 0\;(8) \end{aligned}$$

l, w must be > 0

(6)+(8): Solve for $2lw$ in (6) and (8), set equal: $l^2 - 50 = w^2 - 50$

$\Rightarrow l^2 = w^2$

$\Rightarrow l = w$

Sub into (6): $l^2 + 2l^2 - 50 = 0$

$\Rightarrow 3l^2 = 50$

$\Rightarrow l = \sqrt{\frac{50}{3}}$ $\quad\Rightarrow l = \sqrt{\frac{50}{3}}, \; w = \sqrt{\frac{50}{3}}$

$\Rightarrow$ Critical pts: $\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right)$

Second derivatives test? Instead...

By the physical nature of the problem, there must be an absolute maximum, which also must be a local maximum, and so, must occur at a critical point – but there's only one relevant critical point!

$\Rightarrow f\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right) = \left(\frac{50}{3}\right)^{3/2}$ is an absolute maximum

$\Rightarrow$ The max. volume of a box made from $100\,\text{cm}^2$ of cardboard is $\left(\frac{50}{3}\right)^{3/2}$ cm^3.